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Question	1	2	3	4	5	6	7	8	9	10	Total
Max	14	10	16	10	10	10	10	6	7	7	100
Earned											

Question 1: [14 Points] **[Structural Induction]** [CLO #2] Use structural induction to show that if T is a full binary tree, then $l(T)$ is 1 more than $i(T)$. Where $l(T)$ is the number of leaves of T and $i(T)$ is the number of internal vertices of T.

Leaves are nodes with no children.
Base Step:
(T = leaf).
We have $l(\text{leaf}) = 1 = 1 + 0 = 1 + i(\text{leaf})$
Induction step:
Assume $l(T_1) = 1 + i(T_1)$ and $l(T_2) = 1 + i(T_2)$, where T_1 and T_2 are full binary trees and ($T = T_1 \cdot T_2$).
We have

$l(T_1 \cdot T_2) = l(T_1) + l(T_2)$ by the def. of l
$= 1 + i(T_1) + l(T_2)$ by the IH for T_1
$= 1 + i(T_1) + 1 + i(T_2)$ by the IH for T_2
$= 1 + i(T_1 \cdot T_2)$ by the def. of i

We have completed all the cases for structural induction on a binary tree, so we can therefore conclude that for any full binary tree T, $l(T) = 1 + i(T)$.

Question 2: [10 Points] **[Induction]** [CLO #2] Use mathematical induction to show that $3^n - 1$ is a multiple of 2.

Let $p(n)$ be the proposition: $3^n - 1$ is a multiple of 2 for $n \geq 1$.

Base step: $p(1)$ is true as $3^1 - 1 = 2$ which is multiple of 2.

Inductive step: Assume $p(k)$ is true, i.e., $3^k - 1$ is a multiple of 2 for $k \geq 1$.
We need to show that $3^{k+1} - 1$ is a multiple of 2 for $k \geq 1$.

$$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 + 1) \times 3^k - 1 = (2 \times 3^k) + (1 \times 3^k) - 1$$
$$= (2 \times 3^k) + 3^k - 1$$

(2×3^k) is a multiple of 2 and $3^k - 1$ is a multiple of 2. So $(2 \times 3^k) + 3^k - 1$ is a multiple of 2.

This shows $p(k + 1)$ is true when $p(k)$ is true.

We have completed the base and induction steps; thus, we have shown that $p(n)$ is true for $n \geq 1$.

Question 3: [16 Points] **[Strong Induction]** [CLO #2] Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. Follow this outline to prove that $P(n)$ is true for $n \geq 8$. Complete the missing parts.

(a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are all true, completing the basis step of the proof.
We can write

$8 = 3 + 5$

$9 = 3 + 3 + 3$

$10 = 5 + 5$,

so the statements P(8), P(9), and P(10) are all true.

(b) What is the inductive hypothesis of the proof?

The inductive hypothesis of this proof, since we'll be using strong induction, is that "P(n) is true for all n with $8 \leq n \leq k$, where $k \geq 10$ is some integer." (We take k to be ≥ the biggest base case, which in this case is 10.)

(c) What do you need to prove in the inductive step?

We need to prove that, assuming the inductive hypothesis for k, we have that P(k + 1) is true; that is, that k + 1 can be written as a sum of 3's and 5's.

(d) Complete the inductive step for $k \geq 10$.

We wish to write k + 1 as a sum of 3's and 5's. BY THE INDUCTIVE HYPOTHESIS, P(k − 2) is true (this is why we had to cover 8, 9, and 10 as base cases), letting us write
$k - 2 = 3s + 5r$.
Adding 3 to both sides gives us
$k + 1 = 3s + 5r + 3 = 3(s + 1) + 5r$,
so the P(k + 1) is true.

Since the base cases $n = 8$, $n = 9$, and $n = 10$ all held, and since the inductive step held for $k \geq 10$, $P(n)$ holds for all $n \geq 8$. That is, if n is an integer ≥ 8, we can make a postage of n cents just using 3-cent stamps and 5-cent stamps.

Question 4: [10 Points] **[Probability]** [CLO #3] A car pool contains 12 Fords (4 red and 8 white) and 16 Pontiacs (4 red and 12 white). You are allocated a car at random. You see from a distance that it is red. What is the probability that you have been given a Pontiacs?

Let N be the event of being a Pontiac car

Let R be the event of being a red car

We are asked for P(N|R)

There are 28 cars of which 16 are Pontiacs , so

P(N) = 16/28 = 4/7, and 8 cars are red, so P(R) = 8/28 = 2/7

The probability of a red car, given that it is a Pontiac, is P(R|N) = 4/16 = 1/4

So

$$\mathrm{P(N|R)} = \frac{P(N).P(R|N)}{P(R)} = \frac{(\frac{4}{7})(\frac{1}{4})}{\frac{2}{7}} = \frac{1}{2} = 0.5$$

Question 5: [10 Points] **[Counting & Probability]** [CLO #3] Suppose that 100 people enter a contest and that different winners are selected at random for first, second, third and fourth prizes. What is the probability that Ali, Salem, Ameer, and Ahmad each win a prize if each has entered the contest? Show your answer using combination and division of numbers.

Assume that no person wins more than 1 prize.

(1st,2nd,3rd, 4th)

100, 99 ,98, 97

|S| = 100 x 99 x 98 x 97

|1st| = 4

|2nd| = 3

|3rd| = 2

|4th| = 1

|1st and 2nd and 3rd and 4th| = 10

|W| = 10

P(W) = 10/(100 x 99 x 98 x 97) = 1/(10 x 99 x 98 x 97)

Question 6: [10 Points] **[Pigeonhole Principle]** [CLO #3] There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

The baskets are the pigeons, and we place each of them in one of 24 pigeonholes according to how many apples are in it. Thus the ratio n/k of pigeons to pigeonholes is $\left\lceil\frac{50}{24}\right\rceil = \lceil 2.0833\rceil = 3$. So, by the Generalized Pigeonhole Principle there are at least this many baskets with the same number of apples.

Question 7: [10 Points] **[Binomial Coefficient]** [CLO #3] What is the coefficient of $x^{88}y^{62}$ in the expansion of $(4x - 5y)^{150}$? Show your answer using combination and multiplication of numbers (no need to calculate a final answer).

$$\binom{150}{62}(4x)^{88}(-5y)^{62} = \binom{150}{62}4^{88}(-5)^{62}x^{88}y^{62}$$

$$\binom{150}{62}4^{88}(-5)^{62}$$

Question 8: [6 Points] **[Permutations & Combinations]** [CLO #3] How many bit strings of length 15 have exactly three 0s?

$$C(15,3) = \binom{15}{3} = \frac{15!}{12!\,3!} = 455$$

Question 9: [7 Points] **[Permutations & Combinations]** [CLO #3] How many bit strings of length 8 with exactly two 0's are there for which the 0's are not adjacent?

As we have exactly two 0's we have exactly six ones. We write the six ones with spaces between them to count the number of possible places where 0 can be inserted. 1 1 1 1 1 1

There five places between the 1's, one place before the string, and one place after the string- a total of seven not-adjacent possible places. We want to select two places out of these seven places. So the answer is $C(7,2) = \binom{7}{2} = \frac{7!}{5!2!} = 21$

Question 10: [7 Points] **[Recursive Definition]** [CLO #3] Give a recursive definition of the set of positive integer that are power of 3.

Basis Step: $3 \in S$

Recursive Step: If $x \in S$, then $3x \in S$.